GROUP THEORY 2024 - 25, EXERCISE SHEET 14

Unless stated otherwise, all representations are defined over $\mathbb{C}$, the field of complex numbers. (However most results hold over an arbitrary field as well).

Exercise 1. (hard) *To always do in every course!*
Review the lecture and understand/fill in the gaps in the proofs.

Exercise 2. (easy) *Warm-up*

(1) Let $G = \{e\}$ be the trivial group. Show that representations of G over a field k are in bijective correspondence with vector spaces over k.
(2) For any group G, prove that a one-dimensional representation of G is irreducible.
(3) Show that every group G has an irreducible representation.

Exercise 3. (easy) Let V be a representation of a group G. By $\langle G \cdot v \rangle_{\mathbb{C}}$ we denote the subrepresentation of V generated by $v \in V$. This can be thought of as the smallest subrepresentation of V containing v or the space of all $\mathbb{C}-$ linear combinations of elements of the orbit $G \cdot v$.

Show that V is an irreducible representation of G if and only if $\langle G \cdot v \rangle_{\mathbb{C}} = V$ for all $v \in V - \{0\}$.

Exercise 4. (medium)

(1) Given a finite group G, show that there exists an injective group homomorphism:

$$G \to GL_n(\mathbb{C})$$

where $n = |G|$.

(2) Let V be an irreducible representation of a finite group G. Show that dim $V \leq |G|$.

Note: In fact more is true; $(\text{dim } V)^2 \leq |G|$. But this requires more tools than what we have seen.

Exercise 5. (medium) *Some irreducible representations of S_n*

(1) Find all one-dimensional representations of S_n.

(2) Consider the following vector space:

$$V_n = \{(x_1, ..., x_n) \in \mathbb{C}^n \mid \sum_i x_i = 0\}.$$

(a) Let $e_1, ..., e_n$ be the standard basis of $\mathbb{C}^n$. Find a basis of V_n in terms of this basis.

(b) Show that S_n acts on V_n by permuting the coordinates. This makes V_n into an S_n- representation.
(c) Finally show that V_n is an irreducible S_n representation. This $n-1$ dimensional irreducible representation of S_n is called the standard representation of S_n.

Exercise 6. (medium) *$Hom_{\mathbb{C}}(V, W)$ as a $G-$ representation.*
Recall that given vector spaces V and W, the set of linear maps between V and W, denoted by $Hom_{\mathbb{C}}(V, W)$, is itself a $\mathbb{C}-$ vector space.

(1) Let V, W be representations of a group G. Show that $Hom_{\mathbb{C}}(V, W)$ has the structure of a $G-$ representation with the $G-$ action defined as follows:

$$(g \cdot T)(v) := g \cdot (T(g^{-1} \cdot v))$$

where $g \in G$, $T \in Hom_{\mathbb{C}}(V, W)$ and $v \in V$.

(2) We denote the set of $G-$ intertwiners between V and W as $Hom_{\mathbb{C}[G]}(V, W)$. Also consider the following sub-space of $Hom_{\mathbb{C}}(V, W)$:

$$Hom_{\mathbb{C}}(V, W)^G := \{T \in Hom_{\mathbb{C}}(V, W) \mid g \cdot T = T \text{ for all } g \in G\}.$$

Show that $Hom_{\mathbb{C}[G]}(V, W) = Hom_{\mathbb{C}}(V, W)^G$.

Therefore the space of intertwiners is exactly the sub-representation of $Hom_{\mathbb{C}}(V, W)$ on which G acts trivially.

Exercise 7. (Hard) *Irreducible representations of finite Abelian groups*

(1) Let G be a finite abelian group. Show that all irreducible representations of G are one-dimensional.

(2) What are all the irreducible representations of $\mathbb{Z}/n\mathbb{Z}$?